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Generalized Polynomials of Minimal Norm*

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We consider the problem of finding optimal generalized polynomials of minimal L_p norm $(1 \leqslant p < \infty)$. As an application of our results we obtain Gaussian quadrature formulas for extended Tchebycheff systems.

I. INTRODUCTION

The problem of finding the element of minimal L_p norm $(1 \leqslant p < \infty)$ from a family of generalized polynomials is considered where the multiplicities of the zeros are specified. The zeros themselves are permitted individually to be either fixed or variable. The existence and uniqueness of the "minimal element" is demonstrated. The L_1 case was studied in [4, 5]. In our approach we develop a concise induction process and create a norm improvement technique while separating multiple zeros to dispose of the uniqueness and existence questions. We also note that the existence and uniqueness is routine in the simple zero case, but it becomes a quite difficult problem in the case of multiple zeros because of the non-linearity.

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As an application of our results we obtain Gaussian quadrature formulas for extended Tchebycheff systems.

Because of the factorization properties of polynomials it was well known how to prove the existence and uniqueness of Gaussian quadrature formulas for polynomials via a variational principle. However, it was no clear how to generalize these results to extended Tchebycheff systems. See the interesting discussion in the introduction to [7]. To overcome these difficulties S. Karlin and A. Pinkus in [7] proved the existence and uniqueness of Gaussian quadrature formulas for extended complete Tchebycheff systems but they did not use a variational principle. D. Barrow [2] then gave a very elegant proof of these results for extended Tchebycheff systems again not using this technique. Furthermore, in [8], S. Karlin and A. Pinkus showed the existence of Gaussian quadrature formulas via this principle (see [5] also). We obtain the results entirely via a variational principle. Although the factorization properties of polynomials were not available, we feel our method of proof is no more intricate than the standard proof for polynomials.

II. Existence of a Minimal Solution

Let $\{u_i\}_{i=1}^{N+1} \subset C^N[a, b]$ with $[0, 1] \subset (a, b)$ be a given set of functions with the property that

$$\{u_i\}_{i=1}^{K} \text{ forms an "extended Tchebycheff system" for } K = N-1, N, N+1. \tag{1}$$

Without loss of generality we may assume that

$$\det\{u_i(t_j)_{j=1}^{K}\ {}_{i=1}^{K}\} > 0 \qquad (K = N-1, N, N+1) \tag{2}$$

for each choice of the points, $(t_i)_{i=1}^{K}$, where $a < t_1 \leqslant \cdots \leqslant t_K < b$. Here we adopt the usual convention that if

$$t_{j-1} < t_j = t_{j+1} = \cdots = t_{j+\lambda},$$

then the $(j+\lambda)$th column in the matrix (2) is interpreted as

$$u_1^{(\lambda)}(t_j), ..., u_K^{(\lambda)}(t_j).$$

For $\bar{\tau} = (\tau_1, ..., \tau_N) \in R^N$, we denote by $u(\bar{\tau}; t)$ the unique polynomial of the type

$$u = u_{N+1} + \sum_{i=1}^{N} \alpha_i u_i$$

with real coefficients $\{\alpha_i\}$ such that

$$u(\tau_k)=0, \qquad k=1, ..., N, \tag{3}$$

where a "multiple τ_k" is interpreted as a zero of u with the corresponding multiplicity. (If the value of the kth component of $\bar{\tau}$ is taken on v times by the components of $\bar{\tau}$ we say "τ_k has multiplicity v.")

We are given the multiplicities $\{v_k\}_{k=1}^n$, $\{\mu_k\}_{k=1}^m$ such that $\sum_{k=1}^n v_k+\sum_{k=1}^m \mu_k=N$ and the points

$$\bar{\xi}=(\xi_1, ..., \xi_m) \qquad \text{where} \quad a<\xi_1 \leqslant \cdots \leqslant \xi_m<b.$$

Set $\Delta_n=\{\bar{x}=(x_1, ..., x_n): 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1\}$ and for each $\bar{x}=(x_1, ..., x_n) \in \Delta_n$, let $\bar{\tau}=\bar{\tau}(\bar{x}, \bar{\xi}) \in R^N$ be of the form

$$\bar{\tau}(\bar{x}, \bar{\xi})=(\underbrace{x_1, ..., x_1}_{v_1}, \underbrace{x_2, ..., x_2}_{v_2}, x_3, ..., x_{n-1}, \underbrace{x_n, ..., x_n}_{v_n},$$

$$\underbrace{\xi_1, ..., \xi_1}_{\mu_1}, \underbrace{\xi_2, ..., \xi_2}_{\mu_2}, \xi_3, ..., \xi_{m-1}, \underbrace{\xi_m, ..., \xi_m}_{\mu_m}).$$

Then if

$$I(\bar{x}, \bar{\xi})=\int_0^1 |u(\bar{\tau}(\bar{x}, \bar{\xi}); t)|^p \, dt,$$

where $1 \leqslant p<\infty$, we seek the $\bar{x}^* \in \Delta_n$ with the property

$$I(\bar{x}^*, \bar{\xi}):=\min_{\bar{x} \in \Delta_n} I(\bar{x}, \bar{\xi}) \tag{4}$$

and we refer to it as the

$$\left(\begin{array}{c|c} x_1, ..., x_n & \xi_1, ..., \xi_m \\ v_1, ..., v_n & \mu_1, ..., \mu_m \end{array}\right) \text{problem.} \tag{5}$$

LEMMA 1. *For $\bar{x} \in \Delta_n$ and $k=1, ..., n$ the function*

$$v_k(t)=\frac{\partial}{\partial x_k} u(\bar{\tau}(\bar{x}, \bar{\xi}); t)$$

belongs to the linear span of $\{u_i\}_{i=1}^N=:\langle u_1, ..., u_N\rangle$ and

$$\operatorname{sgn} v_k(t)=\operatorname{sgn} u(\bar{\tau}(\bar{x}, \bar{\xi}); t) \operatorname{sgn}(x_k-t) \qquad \text{a.e. in } [0, 1] \tag{6}$$

$v_k(t)$ has the same zeros including multiplicities as $u(\bar{\tau}(\bar{x}, \bar{\xi}); t)$ except at x_k where the multiplicity is one less. (7)

Proof. Let $\bar{\tau}=\bar{\tau}(\bar{x}, \xi)$. Since $u(\bar{\tau}; t)$ has leading coefficient one, we find $v_k \in \langle u_1, ..., u_N \rangle$. It is easy to check that $u(\bar{\tau}; \cdot)$ can be written in the form

$$u(\bar{\tau}; t)=\det \begin{pmatrix} u_i[\tau_1, ..., \tau_j] & u_i(t) \\ j=1, ..., N & i=1, ..., N+1 \\ i=1, ..., N+1 & \end{pmatrix} \left[\det \begin{pmatrix} u_i[\tau_1, ..., \tau_j] \\ i, j=1, ..., N \end{pmatrix}\right]^{-1}, \quad (8)$$

where $u_i[\tau_1, ..., \tau_j]$ is the $(j-1)$st order divided difference of u_i with respect to the nodes $\{\tau_1, ..., \tau_j\}$. A straightforward computation reveals that v_k satisfies (6) and (7). (For a similar computation see [1]; in particular (6)–(11).)

Remark 1. Let $v_k(t)=c_N u_N+ \cdots +c_1 u_1$. Then using the data at the zeros of v_k and any $t>\max \tau_i$, (2) and Cramer's rule imply

$$\operatorname{sgn} c_N=\operatorname{sgn} v_k(t). \quad (9)$$

Set $\hat{v}_k=v_k(t)/c_N$. By the same reasoning that was employed in Lemma 1, we find $\hat{v}_{kk}:=(\partial/\partial x_k)\, \hat{v}_k(t) \in \langle u_1, ..., u_{N-1} \rangle$ and it has the features:

If τ_k has multiplicity $v_k>1$, then $\hat{v}_{kk}$ has the same zeros including multiplicities as $u(\bar{\tau}(\bar{x}, \xi); t)$ except at x_k where the multiplicity is two less,

$$\hat{v}_{kk} \equiv 0 \qquad \text{if} \quad v_k=1;$$

$$\operatorname{sgn} \hat{v}_{kk}(t)=\operatorname{sgn} \hat{v}_k(t) \operatorname{sgn}(x_k-t) \qquad \text{a.e. on } [0, 1] \text{ if } v_k>1. \quad (10)$$

Combining (6), (9), and (10) we obtain

$$\operatorname{sgn} c_N \hat{v}_{kk}(t)=\operatorname{sgn} u(\bar{\tau}(\bar{x}, \xi); t) \qquad \text{a.e. on } [0, 1] \text{ for } v_k>1. \quad (11)$$

THEOREM 1. *Let $1 \leqslant p < \infty$. Then there is an $\bar{x}^* \in \Delta_n$ which satisfies* (4). *Further any such minimizing $\bar{x}^*=(x_1^*, ..., x_n^*)$ is in the interior of Δ_n (written* int Δ_n), *i.e.*, $0<x_1^*< \cdots <x_n^*<1$.

Proof. From the representation (8), it is clear that $I(\bar{x}, \xi)$ is a continuous function of $\bar{x}$ over Δ_n, a compact set in R^n. Thus there exists a minimizing $\bar{x}^* \in \Delta_n$. We first show that $0<x_1^* \leqslant \cdots \leqslant x_n^*<1$. Say for example that $0=x_1^*$. Let $x_{i_j}^*$ appear $\hat{v}_j$ times as a component of $\bar{x}^*$ where $0=x_{i_1}^*< \cdots <x_{i_l}^* \leqslant 1$ and $\sum_{j=1}^{l} \hat{v}_j=\sum_{i=1}^{n} v_i$. The so-called normal equations are

$$0 \leqslant \frac{\partial I(\bar{x}^*, \xi)}{\partial x_{i_j}^*}=p \int_0^1 |u|^{p-1}(\operatorname{sgn} u)\, v_{i_j}\, dt \qquad (j=1, ..., l) \quad (12)$$

with equality for $j>1$ where $u(t):=u(\bar{\tau}(\bar{x}^*, \bar{\xi}); t)$ and $\partial u/\partial x_{i_j}^* = v_{i_j}$. A consequence of (6) is that $(\operatorname{sgn} u)v_{i_1} < 0$ a.e. on $[0, 1]$. Thus

$$\int_0^1 |u|^{p-1}(\operatorname{sgn} u)\, v_{i_1}\, dt < 0,$$

contradicting (12). Thus $\bar{x}_1^* \in (0, 1)$.

We next show that $\bar{x}^*$ lies in the interior of Δ_n. To avoid some cumbersome notation we assume that $n=2$ and that all the components of $\bar{x}^*$ are equal to x_1^*. The mathematics in the more general case would remain the same. Let $\tau_i = x_1^* + \sqrt{s}\,\mathcal{A}_i$, $i=1, ..., v := v_1 + v_2$ where

$$\mathcal{A}_i = \begin{cases} \dfrac{1}{v_1}, & i=1, ..., v_1 \\ -\dfrac{1}{v_2}, & i=v_1+1, ..., v \end{cases}$$

and set

$$F(s, t) = u(\tau_1, ..., \tau_v, \xi_1, ..., \xi_{N-v}; t).$$

We will establish in this case that

$$\left.\frac{\partial F(s, t)}{\partial s}\right|_{s=0} = -\frac{1}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right)\frac{1}{v(v-1)}\frac{\partial^2 u}{\partial x_1^{*2}}(\bar{\tau}(\bar{x}^*, \bar{\xi}); t) + c\frac{\partial u}{\partial x_1^*}(\bar{\tau}(\bar{x}^*, \bar{\xi}); t) \tag{13}$$

for some constant c. Hence since $F(0, t) = u(\bar{\tau}(\bar{x}^*, \bar{\xi}); t)$

$$\frac{\partial}{\partial s}\left[\int_0^1 |F(s, t)|^p\, dt\right]_{s=0} = p\int_0^1 |F(0, t)|^{p-1}(\operatorname{sgn} F(0, t))\frac{\partial}{\partial s}F(s, t)|_{s=0}\, dt < 0 \tag{14}$$

by (11) and (12) showing that $I(\bar{x}^*, \bar{\xi})$ is not a minimum.

To establish (13) we first expand $F(s, t)$ in a Taylor's series about $s=0$,

$$F(s, t) = u(\bar{\tau}^*, t) + \sum_{i=1}^{v}\left.\frac{\partial u(\bar{\tau}, t)}{\partial \tau_i}\right|_{\bar{\tau}=\bar{\tau}^*}\mathcal{A}_i\sqrt{s} + \frac{1}{2}\sum_{i=1}^{v}\sum_{j=1}^{v}\left.\frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_i\,\partial \tau_j}\right|_{\bar{\tau}=\bar{\tau}^*}\mathcal{A}_i\mathcal{A}_j s + O(s^{3/2}),$$

where $\bar{\tau}^* := \bar{\tau}(\bar{x}^*, \bar{\xi})$.

Since $u(\tau_1, ..., \tau_\nu, \xi_1, ..., \xi_{N-\nu})$ is symmetric in the τ's, the terms in the above expression involving $\sqrt{s}$ vanish; moreover we find

$$\frac{\partial}{\partial s} F(s, t)|_{s=0} = \frac{1}{2}\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)\left(\frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_1^2}\bigg|_{\bar{\tau}=\bar{\tau}^*} - \frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_1\, \partial \tau_2}\bigg|_{\bar{\tau}=\bar{\tau}^*}\right). \tag{15}$$

Next setting $\tau_i = x_1^* + w$, $i = 1, ..., \nu$, it follows that

$$\frac{d}{dw} u(\tau_1, ..., \tau_\nu, \xi_1, ..., \xi_{N-\nu}; t)|_{w=0}$$

$$= \frac{\partial u(\bar{\tau}(\bar{x}^*, \bar{\xi}); t)}{\partial x_1^*} = \nu \frac{\partial u(\bar{\tau}; t)}{\partial \tau_1}\bigg|_{\bar{\tau}=\bar{\tau}^*} \tag{16}$$

$$\frac{d^2}{dw^2} u(\tau_1, ..., \tau_\nu, \xi_1, ..., \xi_{N-\nu}; t)|_{w=0}$$

$$= \frac{\partial^2 u(\bar{\tau}(\bar{x}^*, \bar{\xi}); t)}{\partial x_1^{*2}} = \left[\nu \frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_1^2} + \nu(\nu-1) \frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_1\, \partial \tau_2}\right]_{\bar{\tau}=\bar{\tau}^*}. \tag{17}$$

Considering $\tau_1, ..., \tau_\nu$ distinct at first and then letting them tend to x_1^* yields,

$$\frac{\partial u(\bar{\tau}, t)}{\partial \tau_1} = c_N \hat{v}_1(\tau_2, ..., \tau_\nu, \xi_1, ..., \xi_{N-\nu}; t)$$

and hence

$$\frac{\partial^2 u(\bar{\tau}; t)}{\partial \tau_1^2} = \frac{\partial c_N}{\partial \tau_1} \hat{v}_1(\tau_2, ..., \tau_\nu, \xi_1, ..., \xi_{N-\nu}; t),$$

where as noted $\hat{v}_1$ does not depend on τ_1.

Thus both $\partial u(\bar{\tau}; t)/\partial \tau_1$ and $\partial^2 u(\bar{\tau}; t)/\partial \tau_1^2$ are scalar multiples of $\partial u(\bar{\tau}(\bar{x}^*, \bar{\xi}); t)/\partial x_1^*$ since they all have the same set of zeros including multiplicities, which are $N-1$ in number. Hence it follows from this and (15), (16), (17) that (13) is valid.

III. Uniqueness of the Solution

For the problem defined in (5) we will show that the solution to the normal equations is unique; that is, there exists exactly one $\bar{x}$ in the interior of Δ_n such that

$$f_k(\bar{x}; \bar{\xi}) := \int_0^1 |u(\bar{x}, \bar{\xi}; t)|^{p-2} u(\bar{x}, \bar{\xi}; t)\, v_k(\bar{x}, \bar{\xi}; t)\, dt = 0 \qquad (k = 1, ..., n). \tag{18}$$

(See (12) for $p = 1$.)

Combining this with the results of Theorem 1 will show that there is exactly one solution to the minimization problem in Δ_n and this solution is in int Δ_n.

Let $J = J(\bar{x}, \xi)$ be the Jacobian matrix of $\bar{x}$, a solution to (18) with $\bar{x} \in \text{int}\,\Delta_n$, i.e.,

$$J := \frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)}.$$

LEMMA 2. Det $J > 0$,

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial}{\partial x_j} \int_0^1 |u|^{p-1} \operatorname{sgn} u v_k \, dt$$

$$= (p-1) \int_0^1 |u|^{p-2} v_k v_j \, dt + \int_0^1 |u|^{p-1} \operatorname{sgn} u v_{kj} \, dt + |u|^{p-1} v_k \operatorname{sgn} u \Big|_{x_j^+}^{x_j^-},$$

where $v_{kj} := \partial v_k / \partial x_j$ *and if* $p = 1$, $|u|^{p-2} \equiv 0$.

Proof. It is easy to see that $v_{kl} \in \langle u_1, ..., u_N \rangle$. Note further that for $p > 1$ $|u(x_k^\pm)|^{p-1} = 0$. Thus if $\bar{x}$ satisfies (18):

$$\frac{\partial f_k}{\partial x_j} = (p-1) \int_0^1 |u|^{p-2} v_k v_j \, dt \qquad \text{for} \quad k \neq j \text{ if } p > 1, \tag{19}$$

$$\frac{\partial f_k}{\partial x_j} = 0 \qquad \text{for} \quad k \neq j \text{ if } p = 1. \tag{20}$$

In the case where $k = j$ we find

$$\frac{\partial f_k}{\partial x_k} = (p-1) \int |u|^{p-2} (v_k)^2 \, dt + \int_0^{\cdot} |u|^{p-2} u v_{kk} \, dt \quad \text{for } p > 1 \tag{21}$$

$$\frac{\partial f_k}{\partial x_k} = \int_0^1 \operatorname{sgn} u v_{kk} \, dt = C_N \int_0^1 \operatorname{sgn} u v_{kk} \, dt > 0$$

$$\text{if } p = 1 \text{ and } v_k > 1 \text{ (note (11))} \tag{22}$$

$$\frac{\partial f_k}{\partial x_k} = \int_0^1 \operatorname{sgn} u v_{kk} \, dt + 2 v_k(x_k) \operatorname{sgn} u(x_k^-)$$

$$= C_N \int_0^1 \operatorname{sgn} u v_{kk} \, dt + 2 v_k(x_k) \operatorname{sgn} u(x_k^-)$$

$$> 2 v_k(x_k) \operatorname{sgn} u(x_k^-) > 0 \tag{23}$$

for $p = 1$ and $v_k = 1$ (note (6) and (11)). Combining (19)–(23) yields the result. ▮

In the next result we follow [4].

THEOREM 2. *For each* $n, m \geqslant 0$ *so that* $n+m \leqslant N$, *there is a unique solution* $\bar{x}^* \in \Delta_n$ *to the* $(\bar{x}, \xi)$ *problem cited in* (5). *Furthermore* $\bar{x}^* \in \operatorname{int} \Delta_n$.

Proof. According to our previous remarks it suffices to show that (18) has at most one solution. For $n=1$ the result follows from (23). Let us assume the result is valid for $n-1$ and all m. Hence for each $\xi \in (a, b)$ the problem

$$\begin{pmatrix} x_1, ..., x_{n-1} & \xi_1, ..., \xi_m, \xi \\ v_1, ..., v_{n-1} & \mu_1, ..., \mu_m, v_n \end{pmatrix} \tag{24}$$

has only one solution: $0 < x_1(\xi) < x_2(\xi) < \cdots < x_{n-1}(\xi) < 1$. By the uniqueness and the continuity of Eq. (18), $x_{n-1}(\xi)$ is a continuous function of ξ and hence there is a $\hat{\xi}_1 \in (0, 1)$ so that $\hat{\xi}_1 = x_{n-1}(\hat{\xi}_1)$. Again by induction, the problem

$$\begin{pmatrix} x_1, ..., x_{n-2}, x_{n-1} & \xi_1, ..., \xi_m \\ v_1, ..., v_{n-2}, v_{n-1}+v_n & \mu_1, ..., \mu_m \end{pmatrix} \tag{25}$$

has only one solution. Note that if there were two fixed points $\hat{\xi}_1$ and $\hat{\xi}_2$, according to (18) each would satisfy the normal equations for "problem (24)"; i.e.,

$$f_k(x_1(\hat{\xi}_i), ..., x_{n-1}(\hat{\xi}_i); \xi_1, ..., \xi_m, \hat{\xi}_i) = 0, \qquad k = 1, ..., n-1, i = 1, 2.$$

It is easy to see that both $(x_1(\hat{\xi}_1), ..., x_{n-1}(\hat{\xi}_1))$ and $(x_1(\hat{\xi}_2), ..., x_{n-1}(\hat{\xi}_2))$ satisfy the normal equations corresponding to "problem (25)." But since $x_{n-1}(\hat{\xi}_1) \neq x_{n-1}(\hat{\xi}_2)$, we have a contradiction of the unicity of (25). Thus $\hat{\xi}_1$ is unique and $\xi > \hat{\xi}_1 \Rightarrow x_{n-1}(\xi) < \xi$. For each $\xi > \hat{\xi}_1$, $(x_1(\xi), ..., x_{n-1}(\xi))$ satisfies the equations

$$f_k(x_1(\xi), ..., x_{n-1}(\xi); \xi_1, ..., \xi_m, \xi) = 0 \quad (k = 1, ..., n-1)$$

which is equivalent to

$$f_k(x_1(\xi), ..., x_{n-1}(\xi), \xi; \xi_1, ..., \xi_m) = 0 \qquad (k = 1, ..., n-1). \tag{26}$$

We know by Theorem 1 that there is a $\xi > \hat{\xi}_1$, so that

$$\varphi(\xi) := f_n(x_1(\xi), ..., x_{n-1}(\xi), \xi; \xi_1, ..., \xi_m) = 0.$$

We claim there is exactly one such ξ. Consider any $\xi > \hat{\xi}_1$ with the property $\varphi(\xi) = 0$.

Differentiating,

$$\varphi'(\xi)=\sum_{i=1}^{n-1}\frac{\partial\varphi}{\partial x_i}x_i'(\xi)+\frac{\partial\varphi}{\partial\xi}. \tag{27}$$

Further, (27) and an application of the *implicit function theorem* to (26) yield the n linear equations

$$\sum_{j=1}^{n-1}\frac{\partial\varphi}{\partial x_j}x_j'(\xi)-\varphi'(\xi)=\frac{-\partial\varphi}{\partial\xi}$$

$$\sum_{j=1}^{n-1}\frac{\partial f_i}{\partial x_j}x_j'(\xi)=\frac{-\partial f_i}{\partial\xi}\qquad(i=1,...,n-1)$$

in the n unknowns $(x_1'(\xi), ..., x_{n-1}'(\xi), \varphi'(\xi))$. Solving for $\varphi'(\xi)$ and using Lemma 2

$$\varphi'(\xi)=\det\left(\frac{\partial(f_1, ..., f_{n-1}, \varphi)}{\partial(x_1, ..., x_{n-1}, \xi)}\right)\left[\det\left(\frac{\partial(f_1, ..., f_{n-1})}{\partial(x_1, ..., x_{n-1})}\right)\right]^{-1}>0.$$

Thus there is exactly one solution and the induction has been advanced. ∎

Remark. In the case when $p=1$ and the ν_i and μ_i are all even integers we have

$$I(\bar{x}, \xi)=\int_0^1 u(\bar{x}, \xi; t)\,dt$$

and from our results there is a unique $\bar{x}^*\in\operatorname{int}\Delta_n$ with the properties:

$$0=\int_0^1\frac{\partial u}{\partial x_i^*}(\bar{x}^*, \xi; t)\,dt,\qquad i=1, ..., n. \tag{28}$$

It is easy to show using (28) [7] that there is a unique divided difference quadrature formula of the form

$$\begin{aligned}Q(f)=\sum_{i=1}^{m}\sum_{j=0}^{\mu_i-1}a_{ij}f[\overbrace{\xi_1, ..., \xi_1}^{\mu_1}, \overbrace{\xi_2, ..., \xi_2}^{\mu_2}, ..., \overbrace{\xi_i, ..., \xi_i}^{j+1}]\\+\sum_{i=1}^{n}\sum_{j=0}^{\nu_i-2}b_{ij}f[\overbrace{\xi_1, ..., \xi_1}^{\mu_1}, ..., \overbrace{\xi_m, ..., \xi_m}^{\mu_m}, \overbrace{x_1, ..., x_1}^{\nu_1-1}, ..., \overbrace{x_i, ..., x_i}^{j+1}],\end{aligned} \tag{29}$$

where $\bar{x}=(x_1, ..., x_n)\in\operatorname{int}\Delta_n$ and $\{a_{ij}, b_{ij}\}\subset R$ so that

$$Q(u_i)=\int_0^1 u_i(t)\,dt,\qquad i=1, ..., N.$$

Further the "unique $\bar{x}$" is $\bar{x}^*$. This result was recently obtained by D. L. Johnson [6] and Bojanov *et al.* [5] by other methods. If $\xi_i \in (a, b)\backslash(0, 1)$, the condition that the μ_i are even can be eliminated.

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